

Resilience in consensus dynamics via competitive interconnections

Bahman Gharesifard* Tamer Başar**

* *Coordinated Science Lab, University of Illinois, Urbana-Champaign,
bgharesi@illinois.edu.*

** *Coordinated Science Lab, University of Illinois, Urbana-Champaign,
basar1@illinois.edu.*

Abstract: We show that competitive engagements within the agents of a network can result in resilience in consensus dynamics with respect to the presence of an adversary. We first show that interconnections with an adversary, with linear dynamics, can make the consensus dynamics diverge, or drive its evolution to a state different from the average. We then introduce a second network, interconnected with the original network via an engagement topology. This network has no information about the adversary and each agent in it has only access to partial information about the state of the other network. We introduce a dynamics on the coupled network which corresponds to a saddle-point dynamics of a certain zero-sum game and is distributed over each network, as well as the engagement topology. We show that, by appropriately choosing a design parameter corresponding to the competition between these two networks, the coupled dynamics can be made resilient with respect to the presence of the adversary. Our technical approach combines notions of graph theory and stable perturbations of nonsymmetric matrices. We demonstrate our results on an example of kinematic-based flocking in presence of an adversary.

Keywords: consensus dynamics, competitive networks, distributed control, interconnected systems, saddle-point dynamics, perturbation theory

1. INTRODUCTION

In the past decade, we have witnessed the emergence of networked systems in a variety of interdisciplinary disciplines. These systems enjoy robustness properties, due to the fact that the objective is typically distributed across the individual agents, and they are capable of executing tasks which are global in nature. From an engineering design perspective, considering cooperative interactions within the subsystems appears to be an intuitive approach; and has been a central element of much recent work; see [Bullo et al., 2009a] and references therein. In many systems in nature, however, the interactions are noncooperative or strategic. This phenomenon not only occurs because of the presence of possible adversaries, but often within the subsystems themselves, e.g., [Johnson, 2009]. It is natural then to ask, within a design perspective, whether noncooperative interactions between subsystems following the same basic objective is beneficial. This is the subject of our study in this paper. After establishing a mathematical framework, we demonstrate how competitive interactions can result in achieving resilience in dynamic networks.

Literature review. This paper is related to the literature on consensus dynamics [Olfati-Saber et al., 2007, Ren and Beard, 2008, Bullo et al., 2009b, Mesbahi and Egerstedt, 2010]. The problem of reaching consensus in the presence of an adversary has been recently studied in different contexts including consensus in the presence of misbehaving nodes and failure [Pasqualetti et al., 2012,

Dolev et al., 1986, Zhang and Sundaram, 2012, LeBlanc and Koutsoukos, 2011, Gupta et al., 2006], and robust consensus in delayed-communication [Münz et al., 2010, 2011], in the presence of disturbance and unmodeled dynamics [Hu, 2012], and with dynamically changing interaction topologies and time delays [Chen and Lewis, 2011]. This paper is also related to the literature on interconnected distributed systems [Langbort et al., 2004] and competitive/cooperative interconnections of dynamical systems [Hirsch, 1985]. Finally, parts of our results are inspired by [Gharesifard and Cortés, 2012].

Statement of contributions. We focus on the consensus dynamics on undirected graphs and consider a scenario in which an adversary, with linear dynamics, can corrupt the state estimates of agents by interconnecting with their network. We show that by appropriately choosing its dynamics, the adversary can make the consensus dynamics diverge, or drive its evolution to a state different from the average. Next, we construct a second network, with the same number of vertices, interconnected with the original consensus network via an engagement topology, which itself is a connected graph. The interactions between these two networks is competitive, in the sense that the networks' coupled dynamics correspond to the saddle-point dynamics of a zero-sum game between them. The second network has no information about the adversary, or interactions with it, and has only access to partial information about the state of the consensus network. We show, however, that by appropriately choosing a design parameter corresponding to the competition between these networks, the coupled dynamics can be made resilient with respect to interconnections with the adversary, in the sense that

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the projections of its evolution onto the first component still asymptotically converge to an approximation of the average values. We prove this result by carefully studying the dependence of the set of equilibria to the design parameter and by showing that the problem corresponds to stable perturbations of nonsymmetric matrices. As an auxiliary result, we describe how disconnecting the engagement topology can be used as an intrusion detection mechanism. We demonstrate our results for the kinematic-based flocking in the presence of an adversary.

2. MATHEMATICAL PRELIMINARIES

We start with some notational conventions. Let \mathbb{R} , $\mathbb{R}_{\geq 0}$, \mathbb{Z} , $\mathbb{Z}_{\geq 1}$, $\mathbb{C}_{<0}$ and $\mathbb{C}_{\leq 0}$ denote the set of real, non-negative real, integer, and positive integer numbers, and complex numbers with negative, and nonpositive real parts, respectively. We denote by $|\cdot|$ the Euclidean norm on \mathbb{R}^n , $n \in \mathbb{Z}_{\geq 1}$, and also use the shorthand notation $\mathbf{1}_n = (1, \dots, 1)^T \in \mathbb{R}^n$ and $\mathbf{0}_n = (0, \dots, 0)^T \in \mathbb{R}^n$. We denote the operator norm of a matrix A under the 2-norm by $\|A\|_2$, that is $\|A\|_2 := \max\{\sqrt{\lambda} \mid \lambda \text{ is an eigenvalue of } A^T A\}$. We let I_n and $\mathbf{0}_{n \times n}$ denote, respectively, the identity and zero matrices in $\mathbb{R}^{n \times n}$. For $A \in \mathbb{R}^{n_1 \times n_2}$ and $B \in \mathbb{R}^{d_1 \times d_2}$, $n_1, n_2, d_1, d_2 \in \mathbb{Z}_{\geq 1}$, we let $A \otimes B$ denote their Kronecker product. The function $f : X_1 \times X_2 \rightarrow \mathbb{R}$, with $X_1 \subset \mathbb{R}^{n_1}$, $X_2 \subset \mathbb{R}^{n_2}$ closed and convex, is *concave-convex* if it is concave in its first argument and convex in the second one [Rockafellar, 1997]. A point $(x_1^*, x_2^*) \in X_1 \times X_2$ is a *saddle point* of f if $f(x_1^*, x_2) \leq f(x_1^*, x_2^*) \leq f(x_1, x_2^*)$ for all $x_1 \in X_1$ and $x_2 \in X_2$. Finally, a set-valued map $f : \mathbb{R}^n \rightrightarrows \mathbb{R}^d$ takes elements of \mathbb{R}^n to subsets of \mathbb{R}^d , $n, d \in \mathbb{Z}_{\geq 1}$.

2.1 Graph theory

A *directed graph*, or simply *digraph*, is a pair $\mathcal{G} = (V, E)$, where V is a finite set called the vertex set and $E \subseteq V \times V$ is the edge set. When E is unordered, we call \mathcal{G} an *undirected graph* or simply a *graph*. In this paper, we focus on undirected graphs. If $(u, v) \in E$ is an edge of a graph, we say that u and v are neighbors. A *bipartite graph* is a graph whose vertices can be divided into two disjoint sets V_1 and V_2 such that every edge can be written as (v_1, v_2) or (v_2, v_1) , where $v_1 \in V_1$ and $v_2 \in V_2$. A graph is called *connected* if there exists a path between any two vertices. A *weighted graph* is a triplet $\mathcal{G} = (V, E, A)$, where (V, E) is a graph and $A \in \mathbb{R}_{\geq 0}^{n \times n}$ is the *adjacency matrix* of \mathcal{G} . The adjacency matrix has the property that $a_{ij} > 0$ if $(v_i, v_j) \in E$ and $a_{ij} = 0$, otherwise. Throughout this paper, we assume that $a_{ij} = 1$ if $(v_i, v_j) \in E$. The degree of v_i , $i \in \{1, \dots, n\}$ is $d(v_i) = \sum_{j=1}^n a_{ij}$. The *degree matrix* D is the diagonal matrix defined by $(D)_{ii} = d(v_i)$, for all $i \in \{1, \dots, n\}$. The *Laplacian* is $L = D - A$. For an undirected graph, $L\mathbf{1}_n = \mathbf{1}_n^T L = 0$, $L = L^T$, and L is positive semidefinite [Biggs, 1994]. When \mathcal{G} is connected, the zero eigenvalue is simple.

2.2 Stable perturbations of linear systems

Let us consider the dynamical system Σ with

$$\dot{\mathbf{x}} = A\mathbf{x}(t), \quad (1)$$

where $A \in \mathbb{R}^{n \times n}$, $\mathbf{x}(t) \in \mathbb{R}^n$, for all $t \in \mathbb{R}_{>0}$. A point $\mathbf{x}^* \in X$ is called an *equilibrium point* of (1) if the constant curve $\mathbf{x}(t) = \mathbf{x}^*$, for all $t \in \mathbb{R}_{\geq 0}$, is an evolution of (1).

We call a set $S \subset \mathbb{R}^n$ *stable*, with respect to Σ , if, for any neighborhood U of S , there exists a neighborhood W of S such that all evolution of Σ with initial condition in W remains in U for all subsequent times. S is called *unstable* if it is not stable, and is called *asymptotically stable* if it is stable and also there exists a neighborhood U of S such that all evolution with initial condition in U approaches the set S . We call a matrix A *Hurwitz* if $\text{spec}(A) \subset \mathbb{C}_{<0}$ and *spectrally stable* if $\text{spec}(A) \subset \mathbb{C}_{\leq 0}$. We call a matrix *nondefective* if it has a complete basis of eigenvectors. We next recall from [Kato, 1980, Overton and Womersley, 1988, Burke and Overton, 1992] some stability properties of linear systems under perturbations. Let $A : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$ be given by

$$A(\epsilon) = A_0 + \epsilon A_1, \quad (2)$$

where $A_0, A_1 \in \mathbb{R}^{n \times n}$ and $\epsilon \in \mathbb{R}_{\geq 0}$. Given that A_0 has all its eigenvalues on the imaginary axis and is nondefective, we are interested in finding conditions on A_1 such that $A(\epsilon)$ is spectrally stable and nondefective, when ϵ is small. Unlike symmetric matrices (see Weyl's theorem [Horn and Johnson, 1985, Theorem 4.3.7]), nonsymmetric matrices can in fact be sensitive to such perturbations. Let us now denote the eigenvalues of $A_0 \in \mathbb{R}^{n \times n}$ by $\lambda_1, \dots, \lambda_k$, $k \in \mathbb{Z}_{\geq 1}$ and $k \leq n$, with algebraic multiplicities of $m_1, \dots, m_k \in \mathbb{Z}_{\geq 1}$, and $\sum_{i=1}^k m_i = n$, respectively, where for $i \in \{1, \dots, k\}$, $\text{Re}(\lambda_i) = 0$. Let the diagonalized form of A_0 be given by $J = V A_0 V^{-1}$, where $V \in \mathbb{C}^{n \times n}$, and $J \in \mathbb{C}^{n \times n}$ is a diagonal matrix. The result of [Burke and Overton, 1992, Theorem 7] applies to our situation, as presented next. Note that this result is more general than what we need, since A_0 is allowed to have eigenvalues not only on the imaginary axis but also in the open left-hand plane.

Theorem 1. Let $A(\epsilon)$ be given as in (2), where A_0 is spectrally stable and nondefective, with eigenvalues λ_i , $i \in \{1, \dots, k\}$ and $A_0 = V^{-1} J V$ as above. Let $B = V^{-1} A_1 V$, and let B_i be the diagonal block of B corresponding to λ_i , $i \in \{1, \dots, k\}$. If all eigenvalues of $A(\epsilon)$ have nonpositive real parts for $\epsilon \in (0, \epsilon_0)$, $\epsilon_0 \in \mathbb{R}_{>0}$, then the maximum real parts of the eigenvalues of B_i , $i \in \{1, \dots, k\}$, are less than or equal to zero. Conversely, if the maximum real parts of eigenvalues of B_i , $i \in \{1, \dots, k\}$, are less than or equal to zero, then there exists an interval $(0, \epsilon_0)$ in which $A(\epsilon)$ is spectrally stable and nondefective. \square

3. PROBLEM STATEMENT

Consider a group of $n \in \mathbb{Z}_{\geq 1}$ agents in a connected undirected graph Σ_1 , with the dynamics

$$\dot{\mathbf{x}}(t) = -L\mathbf{x}(t), \quad (3)$$

where $\mathbf{x}(t) = (x_1(t), \dots, x_n(t))^T$, $x_i(t) \in \mathbb{R}$ is the estimate of agent $i \in \{1, \dots, n\}$ at time $t \in \mathbb{R}_{\geq 0}$, and L is the Laplacian of Σ_1 . This dynamics is called the consensus dynamics. It is well-known that when Σ_1 is connected, (3) is asymptotically convergent to the average of the initial values [Olfati-Saber et al., 2007].

Consider next an adversary Σ_{adv} with linear dynamics that can perturb the estimates of agents in Σ_1 by interconnecting with this network as

$$\begin{aligned} \dot{\mathbf{x}} &= -L(\mathbf{x} - \delta \mathbf{y}), \\ \dot{\mathbf{y}} &= P_1 \mathbf{x} - P_2 \mathbf{y}, \end{aligned} \quad (4)$$

where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $P_1, P_2 \in \mathbb{R}^{n \times n}$, and $-P_2$ is assumed to be Hurwitz (the adversary has stable dynamics before

interconnecting with Σ_1). Here $\delta \in \mathbb{R}$ measures how much the network Σ_1 is affected by the signals of the adversary. It is clear that, by appropriately choosing P_1 and P_2 , the adversary can make the dynamics of (4) unstable, or drive its evolution to a state different from the average.

Our objective is to overcome this sensitivity to interconnections with the adversary by interconnecting Σ_1 to a network Σ_2 such that

- (i) the evolution of the dynamics of the interconnected system $\Sigma_{\text{int}} = (\Sigma_1, \Sigma_{\text{adv}}, \Sigma_2)$ from any initial condition are (asymptotically) stable;
- (ii) the projection onto the first component of the evolutions of the dynamics of Σ_{int} from an initial condition ‘approximates’ the evolutions of (3) from the corresponding initial condition.

In this sense, one could potentially think of Σ_2 as a (robust) controller [Başar and Bernhard, 1995]. There are, however, some important features which will distinguish our treatment of this problem from a feedback control design: (1) Σ_2 has no (global) information about the structural properties of the dynamics of Σ_1 and has only access to partial information about the state of Σ_1 , obtained by its agents; (2) Σ_2 has also no access to any information about the adversary and does not even have any knowledge about its presence. In this sense, the main focus of our study will be on identifying certain types of interconnections that naturally inherit robustness. We will elaborate on this further in the next sections.

4. COMPETITIVE INTERCONNECTIONS IN CONSENSUS DYNAMICS

Consider a connected network Σ_2 with the same number of agents as Σ_1 . In this sense, for each agent in Σ_1 there is a corresponding agent in Σ_2 , but the topology of these two networks may not be the same. We denote by \tilde{L} the Laplacian matrix associated with Σ_2 . Agents in Σ_1 and Σ_2 can also obtain information about the estimates of the other network. This is modeled by means of a bipartite connected graph Σ_{eng} , called the *engagement* graph, with disjoint vertex sets $\{v_1, \dots, v_n\}$ in Σ_1 and $\{w_1, \dots, w_n\}$ in Σ_2 . Note that the adjacency matrix A_{eng} associated with Σ_{eng} is of the form $A = \begin{pmatrix} 0_{n \times n} & A_{\text{eng}} \\ A_{\text{eng}} & 0_{n \times n} \end{pmatrix}$, where $A_{\text{eng}} \in \mathbb{R}^{n \times n}$. We then let $L_{\text{eng}} \in \mathbb{R}^{n \times n}$ be the Laplacian associated with A_{eng} , where A_{eng} can be thought of as the adjacency matrix associated with a graph with n vertices, where each v_i is identified with the corresponding w_i , $i \in \{1, \dots, n\}$. In summary, each agent in network Σ_1 obtains information from (i) its neighbors in Σ_1 , (ii) its corresponding agent in Σ_2 and all its other neighbors in Σ_2 according to Σ_{eng} , and vice versa; see Figure 1.

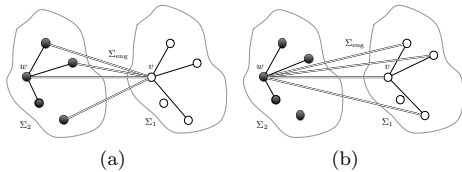


Fig. 1. Two corresponding agents v and w , respectively, in networks Σ_1 and Σ_2 are shown. (a) and (b) show, respectively, the neighbors of v in Σ_2 and the neighbors of w in Σ_1 .

Next, consider the dynamics

$$\begin{aligned} \dot{\mathbf{x}} &= -L\mathbf{x} - \beta L_{\text{eng}}\mathbf{z}, \\ \dot{\mathbf{z}} &= \beta L_{\text{eng}}\mathbf{x} - \tilde{L}\mathbf{z}, \end{aligned} \quad (5)$$

on the interconnected system (Σ_1, Σ_2) , where $\beta \in \mathbb{R}_{>0}$. We make some important observations about the properties of (5). Consider a mapping $U : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ given by

$$U(\mathbf{x}, \mathbf{z}) = \frac{1}{2}\mathbf{x}^T L\mathbf{x} - \frac{1}{2}\mathbf{z}^T \tilde{L}\mathbf{z} + \beta\mathbf{x}^T L_{\text{eng}}\mathbf{z}.$$

Since the graph is undirected, the function U is convex-concave. The next result characterizes the set of saddle points of this function.

Lemma 2. (Saddle points of the mapping U): The set of saddle points of U , for β large enough, is given by $\mathcal{S} = \text{span}\{(\mathbf{1}_n, \mathbf{0}_n), (\mathbf{0}_n, \mathbf{1}_n)\}$.

Clearly, if $(\mathbf{x}, \mathbf{z}) \in \mathcal{S}$, it is a saddle point. The fact that for large values of β , \mathcal{S} characterizes the set of all saddle points, will be proved, as a part of the proof of Proposition 4 later in this section.

Next, consider a static zero-sum game $\mathbf{G} = (\Sigma_1, \Sigma_2, U)$ between Σ_1 and Σ_2 , where Σ_1 wishes to minimize U by choosing $\mathbf{x} \in \mathbb{R}^n$ and Σ_2 wishes to maximize this function by choosing $\mathbf{z} \in \mathbb{R}^n$. The following lemma is an immediate corollary of Lemma 2.

Lemma 3. ((Σ_1, Σ_2) is competitive): The dynamics (5) corresponds to a gradient flow dynamics for seeking a saddle point of the zero-sum game \mathbf{G} . \square

It is important to note that the saddle-point dynamics of a convex-concave function, although stable, does not necessarily need to asymptotically converge to the set of saddle points [Arrow et al., 1958, Feijer and Paganini, 2010]. We, however, establish that an appropriate choice of β makes (5) to asymptotically converge to the set of saddle points of U .

Proposition 4. (Asymptotic stability of (5) for β large): The dynamics of (5), for $\beta \in \mathbb{R}$ large enough, are asymptotically stable.

Proof. It suffices to show that the dynamics $\dot{\mathbf{x}} = A(\epsilon)\mathbf{x}$, with

$$A(\epsilon) = A_0 + \epsilon A_1, \quad (6)$$

where

$$A_0 = \begin{pmatrix} 0_{n \times n} & -L_{\text{eng}} \\ L_{\text{eng}} & 0_{n \times n} \end{pmatrix}, \text{ and } A_1 = \begin{pmatrix} -L & 0_{n \times n} \\ 0_{n \times n} & -\tilde{L} \end{pmatrix}$$

are asymptotically stable for small values of $\epsilon = \frac{1}{\beta} \in \mathbb{R}_{>0}$. We thus need to study the eigenvalues of $A(\epsilon)$ for small values of ϵ . First, note that A_0 has two zero eigenvalues which are not perturbed by A_1 . Using the properties of Kronecker products, one can write $A_0 = F \otimes L_{\text{eng}}$, where $F = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$; thus

$$A_0 = (V_F \otimes V_{L_{\text{eng}}})^{-1} (D_F \otimes D_{L_{\text{eng}}}) (V_F \otimes V_{L_{\text{eng}}}),$$

where $L = V_{L_{\text{eng}}}^{-1} D_{L_{\text{eng}}} V_{L_{\text{eng}}}$, and $F = V_F^{-1} D_F V_F$, where $D_{L_{\text{eng}}} \in \mathbb{R}^{n \times n}$ and $D_F \in \mathbb{R}^{2 \times 2}$ are diagonal matrices corresponding to the diagonalization of L_{eng} and F , respectively, and $V_{L_{\text{eng}}} \in \mathbb{R}^{n \times n}$ and $V_F \in \mathbb{C}^{2 \times 2}$ are invertible. We compute $V_F = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}$; thus $(V_F \otimes V_{L_{\text{eng}}})^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} -iV_{L_{\text{eng}}} & iV_{L_{\text{eng}}} \\ V_{L_{\text{eng}}} & V_{L_{\text{eng}}} \end{pmatrix}$. The inverse of this matrix can be computed as

$$(V_F \otimes V_{L_{\text{eng}}})^{-1} = \sqrt{2} \begin{pmatrix} \frac{1}{2}iV_{L_{\text{eng}}}^{-1} & \frac{1}{2}V_{L_{\text{eng}}}^{-1} \\ -\frac{1}{2}iV_{L_{\text{eng}}}^{-1} & \frac{1}{2}V_{L_{\text{eng}}}^{-1} \end{pmatrix}.$$

A simple calculation then shows that

$$(V_F \otimes V_{L_{\text{eng}}})^{-1}(A_1)(V_F \otimes V_{L_{\text{eng}}}) = \begin{pmatrix} -\frac{1}{2}V_{L_{\text{eng}}}^{-1}(L + \tilde{L})V_{L_{\text{eng}}} & \star \\ \star & -\frac{1}{2}V_{L_{\text{eng}}}^{-1}(L + \tilde{L})V_{L_{\text{eng}}} \end{pmatrix}$$

Since $L + \tilde{L}$ is a Laplacian, thus positive semidefinite with one zero eigenvalue, $-\frac{1}{2}V_{L_{\text{eng}}}^{-1}(L + \tilde{L})V_{L_{\text{eng}}}$ is negative semidefinite (with one zero eigenvalue). Since L_{eng} is symmetric, one can choose $V_{L_{\text{eng}}}$ to be an orthogonal matrix such that $V_{L_{\text{eng}}}^{-1} = V_{L_{\text{eng}}}^T$, yielding that $-\frac{1}{2}V_{L_{\text{eng}}}^{-1}(L + \tilde{L})V_{L_{\text{eng}}}$ is symmetric; thus it has non-positive principal minors. As a result, the conditions of Theorem 1 is satisfied. Furthermore, except for the two zero eigenvalues, there exists $\epsilon_0 \in \mathbb{R}_{>0}$ such that for $\epsilon \in (0, \epsilon_0)$ the rest of eigenvalues all have negative real parts. The fact that the geometric and algebraic multiplicities associated with zero eigenvalues are the same yields that the system is stable [Hirsch and Smale, 1974] and since the only eigenvalues with real parts of zero are these two eigenvalues, any trajectory of (5) asymptotically approaches the set \mathcal{S} . \square

Remark 1. (Relationship to dissipative Hamiltonian dynamics): Note that the dynamics of (5) have similar features with dissipative perturbations of Hamiltonian systems [Maddocks and Overton, 1995], but it does not directly correspond to one such dynamics. This is because A_0 has two zero eigenvalues and is thus degenerate. However, one can show that this dynamics induces a dynamics on $\mathbb{R}^n/\mathbf{1}_n \times \mathbb{R}^n/\mathbf{1}_n$ which indeed is a dissipative Hamiltonian dynamics. \bullet

5. RESILIENCE IN CONSENSUS DYNAMICS VIA COMPETITIVE INTERCONNECTIONS

In this section, we consider the interconnection of dynamics (Σ_1, Σ_2) with the dynamics (5) and study its evolution in the presence of an adversary; see Figure 2. In particular,

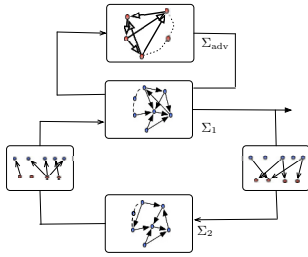


Fig. 2. The interconnected system $(\Sigma_1, \Sigma_2, \Sigma_{\text{adv}})$ is shown. The interconnections between Σ_1 and Σ_2 is distributed, in the sense that is described in Figure 1.

we consider the interconnected system $(\Sigma_1, \Sigma_2, \Sigma_{\text{adv}})$ with the dynamics $\Psi_{A(\beta)} : \mathbb{R}^{3n} \rightarrow \mathbb{R}^{3n}$ given by

$$\underbrace{\begin{pmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{y}} \\ \dot{\mathbf{z}} \end{pmatrix}}_{A(\beta)} = \begin{pmatrix} -L & \delta L & -\beta L_{\text{eng}} \\ P_1 & -P_2 & 0_{n \times n} \\ \beta L_{\text{eng}} & 0_{n \times n} & -\tilde{L} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \end{pmatrix}. \quad (7)$$

We want to know whether $A(\beta)$ is stable when β is chosen large enough. Intuitively, this means that triggering the

competition between agents of Σ_1 and Σ_2 prevents instability imposed by the presence of the adversary. In this sense, one can additionally think that the value of β can be adjusted according to the deviation of \mathbf{x} from some limit. In order to study $A(\beta)$ for β large, equivalently we can consider

$$A(\epsilon) = A_0 + \epsilon A_1, \quad (8)$$

where

$$A_0 = \begin{pmatrix} 0_{n \times n} & 0_{n \times n} & -L_{\text{eng}} \\ 0_{n \times n} & 0_{n \times n} & 0_{n \times n} \\ L_{\text{eng}} & 0_{n \times n} & 0_{n \times n} \end{pmatrix}, \quad A_1 = \begin{pmatrix} -L & \delta L & 0_{n \times n} \\ P_1 & -P_2 & 0_{n \times n} \\ 0_{n \times n} & 0_{n \times n} & -\tilde{L} \end{pmatrix},$$

for small values of $\epsilon = \frac{1}{\beta} \in \mathbb{R}_{>0}$. We show that $A(\epsilon)$ is stable for ϵ small enough. Let us denote by $\text{Ker}(A(\epsilon))$ the kernel of $A(\epsilon)$. We have the following result. The proof is omitted for reasons of space and will appear elsewhere.

Lemma 5. (Properties of the kernel of $A(\epsilon)$): We have

- (i) $\begin{pmatrix} 0 \\ \mathbf{1}_n \end{pmatrix} \in \text{Ker}(A(\epsilon))$, for all $\epsilon \in \mathbb{R}_{>0}$;
- (ii) the mapping $f_{\text{Ker}(A(\cdot))} : \mathbb{R} \rightrightarrows \mathbb{R}^{3n}$, which assigns $\text{Ker}(A(\epsilon))$ to each $\epsilon \in \mathbb{R}_{>0}$, is upper semicontinuous.

Moreover, $\begin{pmatrix} \mathbf{1}_n \\ P_2^{-1}P_1\mathbf{1}_n \\ 0 \end{pmatrix} \in \limsup_{\epsilon \rightarrow 0} \text{Ker}(A(\epsilon))$ and for all $\epsilon \in \mathbb{R}_{>0}$, we have

$$|A(\epsilon) \begin{pmatrix} \mathbf{1}_n \\ P_2^{-1}P_1\mathbf{1}_n \\ 0 \end{pmatrix}| \leq \frac{\epsilon \delta \sqrt{n} \|L\|_2 \|P_1\|_2}{\|P_2\|_2}.$$

- (iii) $\limsup_{\epsilon \rightarrow 0} \text{card}(\text{Ker}(A(\epsilon))) = 2$.

Next, we present our main result.

Theorem 6. (Resilience in consensus against linear adversaries via zero-sum competitive interconnections): Consider the interconnected system $\Sigma_{\text{int}} = (\Sigma_1, \Sigma_{\text{adv}}, \Sigma_2)$ with the dynamics (7). Then for any $P_1, P_2 \in \mathbb{R}^{n \times n}$, where P_2 is symmetric positive definite, there exists $\beta^* \in \mathbb{R}$ such that for all $\beta > \beta^*$ the trajectories of (7) from any initial condition are asymptotically stable. Furthermore, for the initial condition $(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$, if

$$\xi < \frac{\delta \sqrt{n} \|L\|_2 \|P_1\|_2}{\beta^* \|P_2\|_2}, \quad (9)$$

then $|\mathbf{x}^* - \tilde{\mathbf{x}}^*| \leq \xi$ and $|\mathbf{z}^* - \tilde{\mathbf{z}}^*| \leq \xi$, $\xi \in \mathbb{R}_{>0}$, where $(\tilde{\mathbf{x}}^*, \tilde{\mathbf{z}}^*) \in \mathbb{R}^n \times \mathbb{R}^n$ is an equilibrium of (5) with $\mathbf{1}^T \tilde{\mathbf{x}}^* = \mathbf{1}^T \mathbf{x}_0$ and $\mathbf{1}_n^T \tilde{\mathbf{z}}^* = \mathbf{1}_n^T \mathbf{z}_0$.

Proof. We first show that for any such P_1 and P_2 , there exists β such that (7) is asymptotically stable. Equivalently, it suffices to show that the matrix $A(\epsilon)$ given by (8) is stable on an interval $(0, \epsilon_0)$, $\epsilon_0 \in \mathbb{R}_{>0}$. First, note that A_0 is nondefective. Let us show that the conditions of Theorem 1 are satisfied.

First, note that, using basic properties of Kronecker products, one can write $A_0 = F \otimes L_{\text{eng}}$, where $F = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$; thus

$$A_0 = (V_F \otimes V_{L_{\text{eng}}})^{-1} (D_F \otimes D_{L_{\text{eng}}}) (V_F \otimes V_{L_{\text{eng}}}),$$

where $L = V_{L_{\text{eng}}}^{-1} D_{L_{\text{eng}}} V_{L_{\text{eng}}}$, and $F = V_F^{-1} D_F V_F$, where $D_{L_{\text{eng}}}$ and D_F are diagonal matrices, correspond to the diagonalization of L_{eng} and F , respectively. We compute $V_F = \begin{pmatrix} -i & i & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$; thus

$$(V_F \otimes V_{L_{\text{eng}}}) = \begin{pmatrix} -iV_{L_{\text{eng}}} & iV_{L_{\text{eng}}} & 0_{n \times n} \\ 0_{n \times n} & 0_{n \times n} & V_{L_{\text{eng}}} \\ V_{L_{\text{eng}}} & V_{L_{\text{eng}}} & 0_{n \times n} \end{pmatrix}. \quad (10)$$

The inverse of this matrix can be computed as

$$(V_F \otimes V_{L_{\text{eng}}})^{-1} = \begin{pmatrix} \frac{1}{2}iV_{L_{\text{eng}}}^{-1} & 0_{n \times n} & \frac{1}{2}V_{L_{\text{eng}}}^{-1} \\ -\frac{1}{2}iV_{L_{\text{eng}}}^{-1} & 0_{n \times n} & \frac{1}{2}V_{L_{\text{eng}}}^{-1} \\ 0_{n \times n} & V_{L_{\text{eng}}}^{-1} & 0_{n \times n} \end{pmatrix}. \quad (11)$$

Next, using (10) and (11), we conclude

$$(V_F \otimes V_{L_{\text{eng}}})^{-1}(A_1)(V_F \otimes V_{L_{\text{eng}}}) = \begin{pmatrix} \frac{1}{2}V_{L_{\text{eng}}}^{-1}(\mathbf{L} + \tilde{\mathbf{L}})V_{L_{\text{eng}}} & \star & \star \\ \star & \frac{1}{2}V_{L_{\text{eng}}}^{-1}(\mathbf{L} + \tilde{\mathbf{L}})V_{L_{\text{eng}}} & \star \\ \star & \star & -V_{L_{\text{eng}}}^{-1}P_2V_{L_{\text{eng}}} \end{pmatrix}$$

Note that, since $V_{L_{\text{eng}}}$ (chosen to be an orthogonal matrix) is invertible and P_2 is symmetric positive definite, $-V_{L_{\text{eng}}}^{-1}P_2V_{L_{\text{eng}}}$ is negative definite and has negative definite principal minors. Since $\mathbf{L} + \tilde{\mathbf{L}}$ is a Laplacian, thus positive semidefinite with one zero eigenvalue, $-\frac{1}{2}V_{L_{\text{eng}}}^{-1}(\mathbf{L} + \tilde{\mathbf{L}})V_{L_{\text{eng}}}$ is negative semidefinite (with one zero eigenvalue). Since, by Lemma 5, for ϵ small, the kernel of $A(\epsilon)$ is of rank two, a reasoning similar to the one in the proof of Proposition 4 yields the result. The last statement is an immediate corollary of Lemma 5. \square

Our assumption on P_2 is conservative and the result likely holds for $-P_2$ Hurwitz, since $-V_{L_{\text{eng}}}^{-1}P_2V_{L_{\text{eng}}}$ is also Hurwitz and thus can potentially satisfy the conditions of Theorem 1. Also, note that agents in Σ_2 can detect the presence of the adversary by choosing $\beta = 0$ and evaluating the changes in the estimates of Σ_1 received from their neighbors in this network.

6. SIMULATIONS

Consider a network Σ_1 with five agents $\{v_1, \dots, v_5\}$ as shown in Figure 3. Suppose these agents wish to execute a kinematic-based flocking algorithm (see [Lee and Spong, 2007]), such that they achieve a formation in which $x_j - x_i = j - i$, for all $i, j \in \{1, \dots, 5\}$ (this flocking position is consistent, in the sense of [Lee and Spong, 2007]).

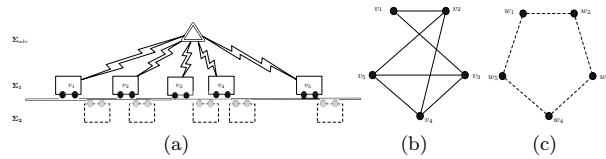


Fig. 3. The formation of agents in the presence of adversary is shown in (a). The agents shown in dashed correspond to Σ_2 . The topologies of Σ_1 and Σ_2 are shown in (b) and (c), respectively. It is assumed here that the topology of Σ_{eng} is similar to Σ_1 in the sense that $L_{\text{eng}} = \mathbf{L}$, where \mathbf{L} is the Laplacian associated with Σ_1 .

The agents can communicate their positions with each other according to the topology shown in Figure 3(b). Since this topology is connected, a consensus dynamics can be used to achieve formation. Now suppose that there is an adversary that can influence the estimates of each agent about the states of its neighbors, according to (4). We consider two cases:

- (i) $P_2 = 3I_5$ and

$$P_1 = \begin{pmatrix} 1.7730 & 1.4254 & 0.0849 & 1.6351 & 1.9459 \\ 0.0573 & 1.0009 & 0.1429 & 1.4449 & 1.2980 \\ 0.9798 & 0.9422 & 1.0433 & 0.2997 & 1.6007 \\ 0.3359 & 0.1192 & 0.1935 & 1.3192 & 0.9076 \\ 1.9574 & 1.3639 & 1.6363 & 1.0372 & 0.8648 \end{pmatrix};$$

- (ii) $P_2 = I_5$ and

$$P_1 = \begin{pmatrix} 3.5833 & 2.4895 & 1.6302 & 0.0931 & 4.0734 \\ 1.4169 & 3.4740 & 2.2821 & 3.3739 & 1.6243 \\ 4.4810 & 4.1718 & 3.5690 & 2.1925 & 1.2311 \\ 4.1329 & 3.0481 & 4.4220 & 2.1891 & 1.7136 \\ 1.9501 & 2.8737 & 3.6043 & 0.5852 & 1.8785 \end{pmatrix}.$$

Figures 4 (a) and (b) show the impact of the signals of the adversary on the stability of the formation dynamics for each case, respectively. In the first scenario, the dynamics of the interconnected system (Σ_1, Σ_2) is asymptotically stable, however, its equilibrium does not correspond to the equidistance formation position without the presence of the adversary. In the second scenario, however, the presence of the adversary causes instability.

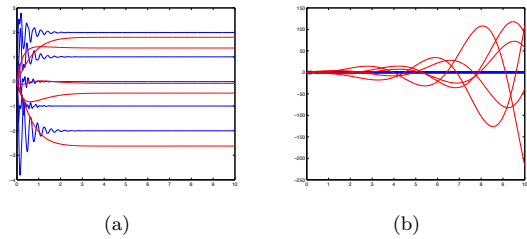


Fig. 4. The position of each agent is shown with respect to time. The adversary is present in both cases, with the dynamics described above. The blue and red plots correspond, respectively, to the cases with and without the presence of Σ_2 . In both cases, all the agents are initially positioned at the origin, $\beta = 10$, $\mathbf{z}(0) = (1, 1, 1, 1, 1)^T$, and $\mathbf{y}(0) = (1, 0, 0, 0, 0)^T$. In (a) $\delta = 0.5$, and in (b) $\delta = 1$.

Suppose a second (virtual) network Σ_2 , with topology shown in Figure 3(c), is interconnected with Σ_1 , where the topology of Σ_{eng} is such that $L_{\text{eng}} = \mathbf{L}$. The interconnected system $(\Sigma_1, \Sigma_{\text{adv}}, \Sigma_2)$ is then asymptotically stable, as shown in Figure 4. In (a), for $\beta = 10$, the final relative position of the agents by

$$\begin{aligned} x_2 - x_1 &= 1.0085, & x_3 - x_2 &= 0.9926, \\ x_4 - x_3 &= 1.0095, & x_5 - x_4 &= 0.9917, \end{aligned}$$

which is significant improvement over the one in the presence of adversary and without Σ_2 :

$$\begin{aligned} x_2 - x_1 &= 2.1572, & x_3 - x_2 &= 0.3947, \\ x_4 - x_3 &= 1.8782, & x_5 - x_4 &= -0.4383. \end{aligned}$$

In (b), for $\beta = 10$, the final relative positions of the agents are given by

$$\begin{aligned} x_2 - x_1 &= 1.0276, & x_3 - x_2 &= 0.9555, \\ x_4 - x_3 &= 1.0023, & x_5 - x_4 &= 1.0501, \end{aligned}$$

whereas the dynamics without Σ_2 is divergent. One can easily verify that (9) holds in both cases.

7. CONCLUSIONS AND FUTURE WORK

We have shown that competition between individuals in a dynamic network can result in resilience with respect to the signals of an adversary with linear dynamics. We have focused on the consensus dynamics on a connected

undirected graph, interconnected with a second network via an engagement topology, corresponding to a zero-sum game. The original network is exposed to corrupting signals of an adversary. The second network has only access to partial information of the original network and has no information, or interconnection, with the adversary. We have shown that, by appropriately choosing a design parameter corresponding to the competition between these two networks, the agents can significantly reduce the impact of the presence of the adversary. Future work will focus on extending the results to systems with nonlinear dynamics, networks with unidirectional topologies, and more general classes of adversaries. We are currently studying self- or event-triggered strategies for increasing competition for recovering from the presence of an adversary, and employing weakening the engagement topologies for detecting possible attacks on a system.

REFERENCES

- K. Arrow, L. Hurwitz, and H. Uzawa. *Studies in Linear and Non-Linear Programming*. Stanford University Press, Stanford, California, 1958.
- T. Başar and P. Bernhard. *H[∞]-Optimal Control and Related Minimax Design Problems*. Birkhäuser, 2 edition, 1995.
- N. Biggs. *Algebraic Graph Theory*. Cambridge University Press, 2 edition, 1994. ISBN 0521458978.
- F. Bullo, J. Cortés, and S. Martínez. Distributed algorithms for robotic networks. In R. A. Meyers, editor, *Encyclopedia of Complexity and System Science*. Springer, New York, 2009a. Entry 00168.
- F. Bullo, J. Cortés, and S. Martínez. *Distributed Control of Robotic Networks*. Applied Mathematics Series. Princeton University Press, 2009b. ISBN 978-0-691-14195-4. Electronically available at <http://coordinationbook.info>.
- J. V. Burke and M. L. Overton. Stable perturbations of nonsymmetric matrices. *Linear Algebra and its Applications*, 171:249–273, 1992.
- G. Chen and F. L. Lewis. Robust consensus of multiple inertial agents with coupling delays and variable topologies. *International Journal on Robust and Nonlinear Control*, 61:666–685, 2011.
- D. Dolev, N. A. Lynch, S. S. Pinter, E. W. Stark, and W. E. Weihl. Reaching approximate agreement in the presence of faults. *Journal of the ACM*, 33:499–516, 1986.
- D. Feijer and F. Paganini. Stability of primal-dual gradient dynamics and applications to network optimization. *Automatica*, 46:1974–1981, 2010.
- B. Gharesifard and J. Cortés. Distributed convergence to Nash equilibria by adversarial networks. *Automatica*, 2012. Submitted. Available at <http://arxiv.org/abs/1204.0852>.
- V. Gupta, C. Langbort, and R. M. Murray. On the robustness of distributed algorithms. In *IEEE Conf. on Decision and Control*, pages 3473–3478, San Diego, CA, December 2006.
- M. W. Hirsch. Systems of differential equations which are competitive or cooperative II: convergence almost everywhere. *SIAM Journal on Mathematical Analysis*, 16:423–439, 1985.
- W. M. Hirsch and S. Smale. *Differential Equations, Dynamical Systems and Linear Algebra*. Academic Press, 1974.
- R. A. Horn and C. R. Johnson. *Matrix Analysis*. Cambridge University Press, 1985. ISBN 0521386322.
- G. Hu. Robust consensus tracking of a class of second-order multi-agent dynamic systems. *Systems & Control Letters*, 61:134–142, 2012.
- L. A. Johnson. Competition interactions between cells: death, growth, and geography. *Science*, 324:1679–1682, 2009.
- T. Kato. *Perturbation theory for linear operators*, volume 132 of *Grundlehren Math. Wiss.* Springer, 1980.
- C. Langbort, R. S. Chandra, and R. D’Andrea. Distributed control design for systems interconnected over an arbitrary graph. *IEEE Transactions on Automatic Control*, 49(9):1502–1519, 2004.
- H. LeBlanc and X. Koutsoukos. Consensus in networked multi-agent systems with adversaries. In *Proceedings of the Hybrid Systems: Computation and Control Conference*, pages 281–290, 2011.
- D. Lee and M. W. Spong. Stable flocking of multiple inertial agents on balanced graphs. *IEEE Transactions on Automatic Control*, 52(8):1469–1475, 2007.
- A. D. Lewis. Semicontinuity of rank and nullity and some consequences. Technical report, Queen’s University, 2009.
- J. Maddocks and M. L. Overton. Stability theory for dissipatively perturbed hamiltonian systems. *Communications on Pure and Applied Mathematics*, XLVIII: 583–610, 1995.
- M. Mesbahi and M. Egerstedt. *Graph Theoretic Methods in Multiagent Networks*. Applied Mathematics Series. Princeton University Press, 2010.
- U. Münz, A. Papachristodoulou, and F. Allgöwer. Delay robustness in consensus problems. *Automatica*, 46: 1252–1265, 2010.
- U. Münz, A. Papachristodoulou, and F. Allgöwer. Robust consensus controller design for nonlinear relative degree two multi-agent systems with communication constraints. *IEEE Transactions on Automatic Control*, 56(1):145–151, 2011.
- E. A. Ok. *Real Analysis with Economic Applications*. Princeton University Press, Princeton, NJ, 2007.
- R. Olfati-Saber, J. A. Fax, and R. M. Murray. Consensus and cooperation in networked multi-agent systems. *Proceedings of the IEEE*, 95(1):215–233, 2007.
- M. L. Overton and R. S. Womersley. On minimizing the spectral radius of a nonsymmetric matrix function: optimality conditions and duality theory. *SIAM Journal on Matrix Analysis and Applications*, 9:473–498, 1988.
- F. Pasqualetti, A. Bicchi, and F. Bullo. Consensus computation in unreliable networks: A system theoretic approach. *IEEE Transactions on Automatic Control*, 57(1):90–104, 2012.
- W. Ren and R. W. Beard. *Distributed Consensus in Multi-vehicle Cooperative Control*. Communications and Control Engineering. Springer, 2008. ISBN 978-1-84800-014-8.
- R. T. Rockafellar. *Convex Analysis*. Princeton Landmarks in Mathematics and Physics. Princeton University Press, Princeton, NJ, 1997. ISBN 0-691-01586-4. Reprint of 1970 edition.
- H. Zhang and S. Sundaram. Robustness of complex networks: reaching consensus despite adversaries. 2012. <http://arxiv.org/abs/1203.6119>.